A festive look at the Szlenk index

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In this talk, we sketch Szlenk's inventive negative solution of this problem.

w^*-open subsets of small diameter

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Let $X = \ell_1$. If U is any non-empty relatively w^*-open subset of $B_{X^*} = B_{\ell_\infty}$, then

$$||\cdot||\text{-diam}\,(U) = 2.$$

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Let $(X, ||\cdot||)$ be a Banach space with $||\cdot||$-separable dual. Now take $\varepsilon > 0$ and a non-empty, w^*-compact subset $A \subseteq X^*$. Then there exists a non-empty, relatively w^*-open subset $U \subseteq A$, such that

$$||\cdot||\text{-diam}\,(U) < \varepsilon.$$

Eating w^*-compact sets

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Second, by Proposition 1, there is a non-empty, relatively w^*-open subset $U \subseteq A$ with $||\cdot||\text{-diam}\,(U) < \varepsilon$, hence

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$$d_\varepsilon(A) \subseteq A \setminus U \subsetneqq A.$$

We can iterate the process:

$$A \supsetneqq d_\varepsilon(A) \supsetneqq d_\varepsilon(d_\varepsilon(A)) \supsetneqq \ldots$$

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In general, if B_α^ε is non-empty for some ordinal α, then $B_{\alpha+1}^\varepsilon = d_\varepsilon(B_\alpha^\varepsilon)$.

And, if B_α^ε is non-empty for all $\alpha < \lambda$, where λ is a limit ordinal, then

$$B_\lambda^\varepsilon \; = \bigcap_{\alpha<\lambda} B_\alpha^\varepsilon.$$

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The *Szlenk index* of X is given by the countable ordinal

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$\mathrm{Sz}(c_0) = Sz(H) = \omega$. For every countable ordinal α, there is a countable compact Hausdorff space K such that $\mathrm{Sz}(C(K)) = \omega^{\alpha+1}$.

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If α is any countable ordinal, then there exists a separable reflexive Banach space X_α such that

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Theorem 13 (Szlenk, 68)

If Y is separable and reflexive, then there is a separable, reflexive space X that does not embed isomorphically in Y.